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AN APPROXIMATION FOR THE DISTRIBUTION OF THE NUMBER OF RETRYING CUSTOMERS IN AN M/G/1 RETRIAL QUEUE

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Abstract. Queueing systems with retrials are widely used to model many problems in call centers, telecommunication networks, and in daily life. We present a very accurate but simple approximate formula for the distribution of the number of retrying customers in the M/G/1 retrial queue.

1. Introduction

Retrial queues are queueing systems in which arriving customers who find all servers occupied may retry for service again after a random amount of time. Retrial queues have been widely used to model many problems in telephone systems, call centers, telecommunication networks, computer networks and computer systems, and in daily life. Detailed overviews for retrial queues can be found in the bibliographies [1, 2, 3], the surveys [6, 9, 10], and the books [4, 7].

Retrial queues are characterized by the following feature: If the server is idle when a customer arrives from outside the system, this customer begins to be served immediately and leaves the system after the service is completed. On the other hand, any customer who finds the server busy upon arrival joins a retrial group, called an orbit, and then attempts to obtain service after a random amount of time. If the server is idle when a customer from the orbit attempts to obtain service, this

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customer receives service immediately and leaves the system after the service completion. Otherwise the customer comes back to the orbit immediately and repeats the retrial process.

We consider the M/G/1 retrial queue where customers arrive from outside the system according to a Poisson process with rate λ and service times are independent and identically distributed. Let B denote a generic random variable representing the service time and $\beta(\cdot)$ be the Laplace-Stieltjes transform of B, i.e., $\beta(s) = \mathbb{E}[e^{-sB}]$. The retrial time, i.e., the length of the time interval between two consecutive attempts made by a customer in the orbit, is exponentially distributed with mean ν^{-1}. The arrival process, the service times, and the retrial times are assumed to be mutually independent. The traffic load ρ is defined as $\rho = \lambda\mathbb{E}B$. We assume that $\rho < 1$ for stability of the system and that the service time distribution has a finite exponential moment, i.e., $\gamma \equiv \sup\{t \in \mathbb{R} : \mathbb{E}e^{tB} < \infty\} > 0$.

At steady state, let N be the number of customers in the orbit (i.e., the number of retrying customers) and C be 0 if the server is idle and 1 otherwise. Let $q_n = \mathbb{P}(N = n, C = 0)$ and $p_n = \mathbb{P}(N = n, C = 1)$, $n = 0, 1, 2, \ldots$. Kim et al. [8] showed that if there exists a real number σ satisfying

$$\beta(\lambda - \lambda\sigma) = \sigma,\ 1 < \sigma < 1 + \frac{\gamma}{\lambda},$$

then

$$q_n \sim cn^{a-1}\sigma^{-n} \quad \text{as } n \to \infty, \tag{1.1}$$

$$p_n \sim \frac{c\nu}{\lambda\sigma} n^a \sigma^{-n} \quad \text{as } n \to \infty, \tag{1.2}$$

where

$$a = \frac{\lambda}{\nu}\frac{\sigma - 1}{-\lambda\beta'(\lambda - \lambda\sigma) - 1},$$

$$c = \frac{1-\rho}{\Gamma(a)}\Big(\frac{\sigma - 1}{\sigma}\Big)^a \exp\Big(\int_1^\sigma \frac{\lambda}{\nu}\frac{1 - \beta(\lambda - \lambda z)}{\beta(\lambda - \lambda z) - z} + \frac{a}{z - \sigma} dz\Big)$$

with $\Gamma(\cdot)$ denoting the gamma function. Here and subsequently, $f_n \sim g_n$ as $n \to \infty$ denotes $\lim_{n\to\infty} \frac{f_n}{g_n} = 1$.

Unfortunately, the constant c is difficult to calculate in practice, with the exclusion of the case when the service time distribution is exponential. Therefore, we present a very accurate but simple approximate formula for the computation of the distributions q_n and p_n. The approximation is based on the tail asymptotics (1.1) and (1.2).

2. Approximation for the distribution of the number of retrying customers

Note that for every positive constant b, we have $n \sim n+b$ as $n \to \infty$. Therefore, (1.1) and (1.2) imply that

$$q_n \sim c(n+b)^{a-1}\sigma^{-n} \quad \text{as } n \to \infty,$$
$$p_n \sim \frac{c\nu}{\lambda\sigma}(n+b)^{a}\sigma^{-n} \quad \text{as } n \to \infty,$$

for $b > 0$. Since the constant c is difficult to calculate in practice, we will use approximations $\tilde{q}_n$ and $\tilde{p}_n$ for q_n and p_n, $n = 0, 1, 2, \ldots$, as shown below: For positive real numbers $\tilde{b}$ and $\tilde{c}$,

$$\tilde{q}_n = \tilde{c}(n+\tilde{b})^{a-1}\sigma^{-n}, \tag{2.1}$$

$$\tilde{p}_n = \frac{\tilde{c}\nu}{\lambda\sigma}(n+\tilde{b})^{a}\sigma^{-n}. \tag{2.2}$$

To use this approximation, we have to determine $\tilde{b}$ and $\tilde{c}$. The $\tilde{q}_n$ and $\tilde{p}_n$ satisfy the following two equations:

$$\sum_{n=0}^{\infty}(\tilde{q}_n + \tilde{p}_n) = 1, \tag{2.3}$$

$$\sum_{n=0}^{\infty}\tilde{p}_n = \rho, \tag{2.4}$$

where (2.3) follows from the condition that the total probability is 1 and (2.4) follows from the fact that $\sum_{n=0}^{\infty}\tilde{p}_n$ is equal to the probability of the server being busy.

We introduce the Lerch transcendent given by (see Section 1.11 of [5])

$$\Phi(z, s, \alpha) = \sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^s}, \ |z| < 1, \alpha \neq 0, -1, -2, \ldots.$$

Substituting (2.1) and (2.2) into (2.3) and (2.4) yields

$$\tilde{c}\Phi(\sigma^{-1}, 1-a, \tilde{b}) + \frac{\tilde{c}\nu}{\lambda\sigma}\Phi(\sigma^{-1}, -a, \tilde{b}) = 1,$$
$$\frac{\tilde{c}\nu}{\lambda\sigma}\Phi(\sigma^{-1}, -a, \tilde{b}) = \rho.$$

From this we have

$$\lambda\sigma\rho\Phi(\sigma^{-1}, 1-a, \tilde{b}) - (1-\rho)\nu\Phi(\sigma^{-1}, -a, \tilde{b}) = 0, \tag{2.5}$$

	Example 1	Example 2
σ	1.697224362268005	1.338562172233852
a	1.470725343394151e-01	5.944911182523071e-02
$\tilde{b}$	1.218698140527216e-01	6.242138657358584e-02
$\tilde{c}$	4.839504546889740e-02	2.229981192806169e-02

TABLE 1. The values of σ, a, $\tilde{b}$ and $\tilde{c}$.

and

$$\tilde{c} = \frac{1-\rho}{\Phi(\sigma^{-1}, 1-a, \tilde{b})}. \tag{2.6}$$

Therefore, the value $\tilde{b}$ is calculated by numerically solving equation (2.5) and the value $\tilde{c}$ is given by (2.6).

In summary, the approximations $\tilde{q}_n$ and $\tilde{p}_n$ for q_n and p_n, $n = 0, 1, 2, \ldots$ are calculated as follows:

$$\tilde{q}_n = \tilde{c}(n+\tilde{b})^{a-1}\sigma^{-n},$$
$$\tilde{p}_n = \frac{\tilde{c}\nu}{\lambda\sigma}(n+\tilde{b})^{a}\sigma^{-n},$$

where σ is a solution of $\beta(\lambda - \lambda\sigma) = \sigma$, $a = \frac{\lambda}{\nu}\frac{\sigma-1}{-\lambda\beta'(\lambda-\lambda\sigma)-1}$, $\tilde{b}$ is calculated by numerically solving equation $\lambda\sigma\rho\Phi(\sigma^{-1}, 1-a, \tilde{b}) - (1-\rho)\nu\Phi(\sigma^{-1}, -a, \tilde{b}) = 0$ and $\tilde{c}$ is given by $\tilde{c} = \frac{1-\rho}{\Phi(\sigma^{-1}, 1-a, \tilde{b})}$.

3. Numerical examples

Numerical examples are presented to illustrate the accuracy of the approximate formulas (2.1) and (2.2). In the following two examples, we assume that the arrival rate is $\lambda = 1$ and the mean service time is $\mathbb{E}B = \frac{3}{2}$ and so the traffic load is $\rho = \frac{2}{3}$. The retrial rate is $\nu = 10$.

EXAMPLE 3.1. *(The $M/E_2/1$ retrial queue). We consider the $M/E_2/1$ retrial queue where the service time distribution is Erlang of order 2 with density function $f(x) = 9xe^{-3x}$.*

EXAMPLE 3.2. *(The $M/H_2/1$ retrial queue). We consider the $M/H_2/1$ retrial queue where the service time distribution is hyperexponential of order 2 with density function $f(x) = \frac{1}{4}(\frac{3}{4}e^{-\frac{3}{4}x}) + \frac{3}{4}(\frac{9}{4}e^{-\frac{9}{4}x})$.*

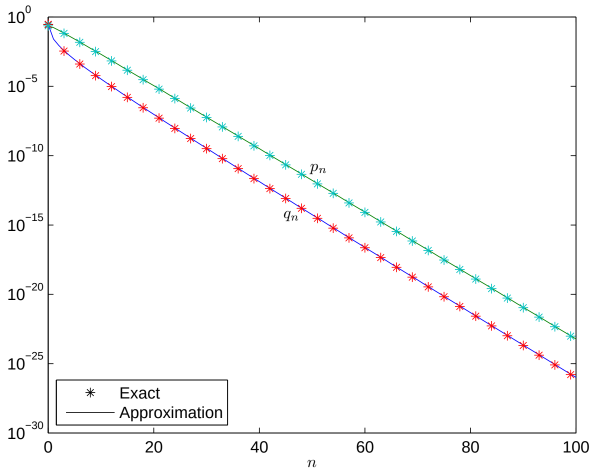

FIGURE 1. Exact and approximate values of q_n and p_n for Example 3.1.

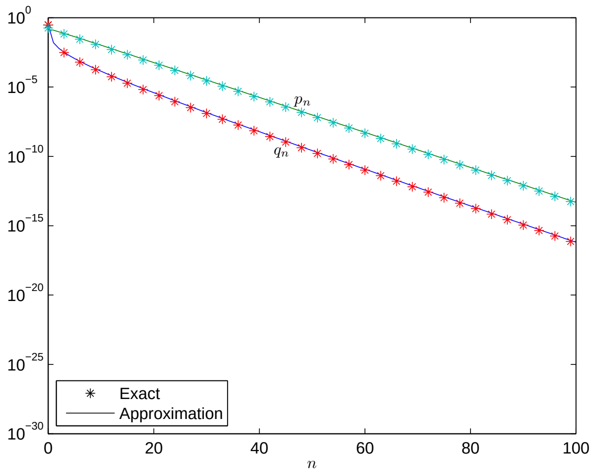

FIGURE 2. Exact and approximate values of q_n and p_n for Example 3.2.

In Figure 1 (Figure 2, resp.), we plot the exact and approximate values of q_n and p_n for Example 3.1 (Example 3.2, resp.). The approximate values are obtained by using the formulas (2.1) and (2.2), along with the

n	q_n^*	$\tilde{q}_n$	rel. error	p_n^*	$\tilde{p}_n$	rel. error
0	2.953e-01	2.914e-01	-1.341e-02	2.297e-01	2.092e-01	-8.917e-02
9	5.942e-05	6.284e-05	5.766e-02	3.203e-03	3.378e-03	5.443e-02
18	2.829e-07	2.995e-07	5.852e-02	3.025e-05	3.197e-05	5.698e-02
27	1.716e-09	1.817e-09	5.888e-02	2.745e-07	2.903e-07	5.787e-02
36	1.150e-11	1.218e-11	5.907e-02	2.449e-09	2.592e-09	5.832e-02
45	8.137e-14	8.619e-14	5.919e-02	2.165e-11	2.291e-11	5.859e-02
54	5.963e-16	6.316e-16	5.927e-02	1.902e-13	2.014e-13	5.877e-02
63	4.475e-18	4.740e-18	5.933e-02	1.665e-15	1.763e-15	5.891e-02
72	3.418e-20	3.621e-20	5.938e-02	1.453e-17	1.539e-17	5.900e-02
81	2.646e-22	2.803e-22	5.941e-02	1.265e-19	1.340e-19	5.908e-02
90	2.070e-24	2.193e-24	5.944e-02	1.099e-21	1.164e-21	5.914e-02
99	1.633e-26	1.730e-26	5.946e-02	9.539e-24	1.010e-23	5.919e-02

Table 2. The values of relative error for Example 3.1. The q_n^* and $\tilde{q}_n$ (p_n^* and $\tilde{p}_n$, resp.) denote the exact and approximate values of q_n (p_n, resp.).

n	q_n^*	$\tilde{q}_n$	rel. error	p_n^*	$\tilde{p}_n$	rel. error
0	3.014e-01	3.029e-01	5.222e-03	1.798e-01	1.413e-01	-2.141e-01
9	1.753e-04	2.033e-04	1.602e-01	1.185e-02	1.377e-02	1.623e-01
18	6.599e-06	7.705e-06	1.676e-01	8.901e-04	1.040e-03	1.660e-01
27	3.265e-07	3.819e-07	1.693e-01	6.601e-05	7.719e-05	1.695e-01
36	1.806e-08	2.113e-08	1.701e-01	4.864e-06	5.692e-06	1.702e-01
45	1.061e-09	1.242e-09	1.706e-01	3.571e-07	4.181e-07	1.707e-01
54	6.478e-11	7.585e-11	1.709e-01	2.616e-08	3.063e-08	1.710e-01
63	4.062e-12	4.757e-12	1.711e-01	1.913e-09	2.241e-09	1.712e-01
72	2.597e-13	3.041e-13	1.713e-01	1.398e-10	1.637e-10	1.713e-01
81	1.685e-14	1.974e-14	1.714e-01	1.020e-11	1.195e-11	1.714e-01
90	1.106e-15	1.296e-15	1.715e-01	7.441e-13	8.717e-13	1.715e-01
99	7.329e-17	8.587e-17	1.716e-01	5.424e-14	6.355e-14	1.716e-01

Table 3. The values of relative error for Example 3.2.

values of σ, a, $\tilde{b}$ and $\tilde{c}$ given in Table 1. The exact values are obtained as follows: It is known that $\lim_{K\to\infty} q_n^{(k)} = q_n$ ($\lim_{K\to\infty} p_n^{(k)} = p_n$, resp.), where $q_n^{(K)}$ ($p_n^{(K)}$, resp.) is the probability that there are n customers in the orbit and the server is idle (busy, resp.) at steady state in the corresponding retrial queue with finite orbit capacity K. The probability q_n

(p_n, resp.) is obtained as $q_n^{(K)}$ ($p_n^{(K)}$, resp.) such that $q_n^{(K)}$ ($p_n^{(K)}$, resp.) does not vary numerically as K increases. Figures 1 and 2 show that the approximations (2.1) and (2.2) are very accurate.

To illustrate the accuracy of our approximation method, we consider the values of relative error. The relative error is defined by $\frac{v_{approx}-v}{v}$, where v is the exact value and v_{approx} is its approximation. The values of relative error for Examples 3.1 and 3.2 are shown in Tables 2 and 3.

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